AST ALGORITHMS OF ASYMPTOTIC ANALYSIS OF NETWORKS WITH UNRELIABLE EDGES

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A problem of a reliability in networks with unreliable elements naturally origin in technical applications [1]. But a direct calculation of the reliability demands a number of operations which increases geometrically dependently on a number of edges. So it is necessary to use approximate methods and particularly asymptotic one. In [2] a reliability asymptotic is calculated in analogous asymptotic suggestions on the network edges. Main parameters in these asymptotic are a shortest way length and a maximal flow in a network. In this paper different partial classes of networks are considered and effective algorithms of their parameters calculations are suggested. These networks are networks originated by dynamic systems, networks with integer-valued lengths of edges, superposition of networks and bridge schemes.

1. Preliminaries

Define the graph $\Gamma$ with the finite nodes set $U$ and the set $W$ of edges $w=(u,v)$. The graph $\Gamma$ may contain cycles or not, its edges may be oriented or not. Denote by $R(u)$ the set of all ways $R$ of the graph $\Gamma$, which connect the nodes $u_0, u$, and assume that $R(u) \neq \emptyset$, $u \in U$. Suppose that $\Gamma(u)$ is the sub-graph of the graph $\Gamma$, which consists of the ways $R \in R(u)$. Consider the sets

$$A(u) = \{ A \subseteq U : u_0 \in A, u \notin A \}, \quad L(A) = \{ (u,u') : u \in A, u' \notin A \}$$

And the set $L(u) = \{ L(A), A \in A(u) \}$ of all sections of the sub-graph $\Gamma(u)$.

Characterize each edge $w \in W$ of the graph $\Gamma$ by the logic number $\alpha(w) = I$ (the edge $w$ works), where $I(B)$ is the indicator function of the event $B$. Denote

$$\beta(u) = \bigvee_{R \in R(u)} \bigwedge_{w \in R} \alpha(w)$$

the characteristic of the nodes $u_0, u$ connectivity in the graph $\Gamma$. Suppose that $\alpha(w), \ w \in W$, are independent random variables, $P(\alpha(w) = 1) = p_w(h), \ q_w(h) = 1 - p_w(h)$, where $h$ is small parameter: $h \to 0$. In [2] the following statements are proved.

**Theorem 1.** Suppose that $p_w(h) \sim \exp(-h^{-c(w)})$, $h \to 0$, where $c(w) > 0, \ w \in W$. Then
\[-\ln P(\beta(u)=1) \sim h^{-D(u)}, \quad D(u) = \min_{R \in R(u)} \max_{w \in R} c(w). \quad (1)\]

**Theorem 2.** Suppose that \( q_w(h) \sim \exp\left(-h^{-c_1(w)}\right), \ h \to 0, \) where \( c_1(w) > 0, \ w \in W. \) Then

\[-\ln P(\beta(u)=0) \sim h^{-D_1(u)}, \quad D_1(u) = \max_{w \in R} c_1(w). \quad (2)\]

**Theorem 3.** Suppose that \( p_w(h) \sim h^{g(w)}, \ h \to 0, \) where \( g(w) > 0, \ w \in W. \) Then

\[-\ln P(\beta(u)=1) \sim T(u) \ln h, \quad T(u) = \min_{R \in R(u)} \sum_{w \in R} g(w). \quad (3)\]

**Theorem 4.** Suppose that \( q_w(h) \sim h^{g(w)}, \ h \to 0, \) where \( g(w) > 0, \ w \in W. \) Then

\[-\ln P(\beta(u)=0) \sim T_1(u) \ln h, \quad T_1(u) = \min_{L \in c(u)} \sum_{w \in L} g(w). \quad (4)\]

**Statement 1.** Suppose that all \( c(w) \) (all \( c_1(w) \)), \( w \in W, \) are different. Then there is the single edge \( w(u) \) (there is the single edge \( w_1(u) \)), so that \( c(w(u)) = D(u) \) \( (c_1(w_1(u)) = D_1(u)). \) It is called the critical edge.

### 2. Graphs generated by dynamic systems

Suppose that the set \( U \) consists of non-intersected subsets \( U_0, U_1, \ldots, U_m, \) and the set \( U_0 \) contains the single vertex \( u_0, \) which is called initial. All edges of the oriented graph \( \Gamma \) are represented as \( (u_i, u_j), 1 \leq i < j \leq m, \ u_i \in U_i, \ u_j \in U_j, \) and each vertex is accessible from the initial vertex \( u_0. \) Described graphs are generated by dynamic systems with a delay. In this section we calculate \( D(u), D_1(u), T(u) \) and find critical edges \( w(u), w_1(u) \) for a fixed \( u_0. \)

A main idea of this section is an application of the Floyd algorithm [3], when a solution is calculated for all \( u \in U. \) To construct fast algorithms it is natural to construct a class of considered graphs. An idea of such a construction is illustrated in [4] but for a fixed \( u. \)

Suppose that \( D(u_0) = D_1(u_0) = T(u_0) = 0, \) for all \( u \in U \) put

\[D(u) = D_1(u) = T(u) = c(u), \quad w(u) = w_1(u) = (u_0, u).\]

For \( u \in U \) define \( S(u) = \{v: (v, u) \in W\}, \ |S(u)| \) a number of elements in the finite set \( S(u). \) Assume that for all \( u \in U_1, \ldots, U_k \) the meanings \( D(u), \ D_1(u), \ T(u), w(u), \ w_1(u) \) are defined. Take \( u \in U_{k+1} \) and in an accordance with the formulas (1), (2) put

\[D(u) = \min_{v \in S(u)} \max\{c(v, u), D(v)\}, \quad D_1(u) = \max_{v \in S(u)} \min\{c(v, u), D_1(v)\}, \quad (5)\]
To calculate each element from the set $D(u)$, $D_1(u)$, $T(u)$, $u \in U$ it is necessary $2|S(u)| - 1$ arithmetic operations and this number can not be decreased. So the algorithm (5), (6) is optimal. And if for fixed $u \in U$ $D(u)$, $D_1(u)$, $T(u)$ are calculated by the algorithm (5), then we find $D(v)$, $D_1(v)$, $T(v)$ for all nodes $v$ from which the vertex $u$ is accessible.

To define critical edges it is necessary to complement the formulas (5) by

$$w(u) = w_1(u) = (u_0, u), \text{ if } u_0 \in S(u),$$

$$w(u) = \begin{cases} w(v), & \text{if } D(u) = \max(D(v), c(v, u)) > c(v, u), \\ (v, u), & \text{if } D(u) = \max(D(v), c(v, u)) > D(v), \end{cases}$$  \hspace{1cm} (7)

$$w_1(u) = \begin{cases} w_1(v), & \text{if } D_1(u) = \max(D_1(v), c(v, u)) < c(v, u), \\ (v, u), & \text{if } D_1(u) = \max(D_1(v), c(v, u)) < D(v). \end{cases}$$  \hspace{1cm} (8)

3. Graphs with integer-valued lengths of edges

In this section we consider a calculation of $T(u)$ for all $u \in U$ in graphs with integer-valued lengths of edges. Suppose that $g(w)$, $w \in W$, are natural numbers, $g(w) \leq g < \infty$ and define

$$G_\Gamma = \sum_{w \in W} g(w).$$  \hspace{1cm} (9)

Divide each edge of the graph $\Gamma$ into edges with unit lengths by an introduction of intermediary nodes. As a result obtain the graph $\Gamma^1$ with the nodes set $U^1$, $U \subseteq U^1$ and with the edges set $W^1$. Denote $N(u^1)$ the minimal number of the graph edges in ways, which connect the nodes $u_0, u^1$. It is easy to obtain that

$$N(u) = G(u), \quad u \in U^1. \hspace{1cm} (10)$$

Consider now an algorithm of $N(u^1)$, $u^1 \in U^1$ calculation.

Suppose that all nodes of the graph $\Gamma^1$ are not marked. Mark the vertex $u_0$, and put $U^1_0 = \{u_0\}$. Then construct a recurrent procedure of non-intersected sets $U^1_k$, $k \geq 0$, definition. Suppose that the sets $U^1_k$, $V^1_k = \bigcup_{0 \leq i \leq k} U^1_i$ are known and all nodes of the set $V^1_k$ are marked and all other nodes are not marked. Define the set $U^1_{k+1}$ as a set of all unmarked nodes from $U^1$, which are
connected directly with some vertex from the set $U^1_k$. By a definition the set $U^1_{k+1}$ satisfies the formula

$$U^1_{k+1} = \bigl\{ u^1 : N\bigl(u^1\bigr) = k + 1 \bigr\}.$$  

Mark all nodes of the set $U^1_{k+1}$ and define the set $V^1_{k+1} = V^1_k \cup U^1_{k+1}$.

Estimate a number of operations which are necessary to calculate $U^1_{k+1}$ if each vertex of the graph $\Gamma$ is connected directly with no more $l$ nodes. Then a number of operations to define $U^1_{k+1}$ does not exceed $l|U^1_k|$. Define $M$ by the formula

$$V^1_0 \subseteq V^1_1 \subseteq \ldots \subseteq V^1_M = V^1_{M+1} = \ldots,$$

then to construct the sequence $U^1_1, \ldots, U^1_M$ it is necessary no more $lG_\Gamma$ operations where $lG_\Gamma \leq l^2g |U|$. Compare these results with the results of Deikstra [4], in a case when $c(w)$ is not integer-valued. To calculate $D(u)$, $u \in U$ in a general case it is necessary no more $K_1|U|^2$ operations and for a dendriform graph - no more $K_2|U|\ln|U|$ operations, where $K_1, K_2 < \infty$.

4. Superposition of graphs

Fix in the graph $\Gamma$ some vertex $v_0$. Assume that $\Gamma'$ is non-oriented graph with the nodes set $U' = \{1', \ldots, m'\}$, $U \cap U' = \emptyset$ and with the edges set $W' = \{(i', j'), (i', i) \in W'\}$. Distinguish in the graph $\Gamma'$ initial and final nodes $u_0', v_0'$ and in the set $U' -$ two nodes $\bar{u}, \bar{v}$ so that $w = (\bar{u}, \bar{v}) \in W$. Denote by $R'$ the set of all ways $R'$ of the graph $\Gamma'$ from $u_0'$ to $v_0'$.

Define the superposition $\Gamma = \Gamma \otimes \Gamma'$ of the graphs $\Gamma, \Gamma'$ with a replacement of the edge $(\bar{u}, \bar{v})$ from the graph $\Gamma$ by the graph $\Gamma'$ and with an aliasing of the nodes $\bar{u}$ with $u_0'$ and of the nodes $\bar{v}$ with $v_0'$ correspondingly. Denote by $\bar{U}$ the nodes set, by $\bar{W}$ - the edges set and by $\bar{R}$ - the set of ways from the vertex $u_0$ to the vertex $v_0$ in the graph $\Gamma$. Put $R$ the set of ways from $u_0$ to $v_0$ in the graph $\Gamma$, $R'$ - the set of ways from $u_0'$ to $v_0'$ in the graph $\Gamma'$. Analogously define $\bar{L}, \bar{L}, \bar{L}'$ the sets of sections in the graphs $\bar{\Gamma}, \Gamma, \Gamma'$ with pairs of initial and final nodes $(\bar{u}_0, \bar{v}_0), (u_0, v_0)$, $(u_0', v_0')$ correspondingly. Define

$$\beta = \bigvee_{R \in \bar{R}, w \in \bar{W}} \wedge_{\bar{R} \in \bar{R}} \alpha(w), \quad \bar{\beta} = \bigvee_{\bar{R} \in \bar{R}} \bigwedge_{w \in \bar{W}} \alpha(w)$$

characteristics of a connectivity between the nodes $u_0, v_0$ in the graphs $\Gamma, \bar{\Gamma}$ correspondingly. Then from the theorems 1-4 it is possible to obtain asymptotic formulas for the superposition $\bar{\Gamma}$.

**Theorem 5.** Suppose that $p_w(h) \sim \exp\left(-h^{-c(w)}\right)$, $h \to 0$, where $c(w) > 0$, $w \in \bar{W}$. Then

$$-\ln P\left(\bar{\beta} = 1\right) \sim h^{-\bar{\beta}}, \quad D = \min_{R \in \bar{R}} \max_{w \in \bar{W}} \bar{c}(w).$$
\( \overline{c}(w) = c(w), \; w \neq \overline{w}, \; \overline{c}(\overline{w}) = \min_{R' \in R''} \max_{w \in R'} c(w). \)

**Theorem 6.** Suppose that \( q_w(h) \sim \exp \left( -h^{c_1}(w) \right), \; h \to 0, \) where \( c_1(w) > 0, \; w \in \overline{W}. \) Then

\[
\ln P\left( \overline{\beta} = 0 \right) \sim h^{D_1}, \; \overline{D_1} = \min_{L \in L'} \max_{w \in R} c_1(w),
\]

\[
\overline{c}_1(w) = c_1(w), \; w \neq \overline{w}, \; \overline{c}_1(\overline{w}) = \min_{L' \in L''} \max_{w \in R'} c_1(w).
\]

**Theorem 7.** Suppose that \( p_w(h) \sim h^{g_1(w)}, \; h \to 0, \) where \( g_1(w) > 0, \; w \in \overline{W}. \) Then

\[
\ln P\left( \beta = 1 \right) \sim \overline{T} \ln h, \; \overline{T} = \min_{R \in R'} \sum_{w \in R} g(w),
\]

\[
\overline{g}(w) = g_1(w), \; w \neq \overline{w}, \; \overline{g}(\overline{w}) = \min_{L' \in L''} \sum_{w \in L'} g_1(w).
\]

**Theorem 8.** Suppose that \( q_w(h) \sim h^{g_1(w)}, \; h \to 0, \) where \( g_1(w) > 0, \; w \in \overline{W}. \) Then

\[
\ln P\left( \overline{\beta} = 0 \right) \sim \overline{T}_1 \ln h, \; \overline{T}_1 = \min_{L \in L' \in L''} \sum_{w \in L'} g_1(w),
\]

\[
\overline{g}_1(w) = g_1(w), \; w \neq \overline{w}, \; \overline{g}_1(\overline{w}) = \min_{L' \in L''} \sum_{w \in L'} g_1(w).
\]

It is obvious that the formulas from these theorems allow calculating asymptotic of a reliability for superposition of networks with unreliable elements rationally. These formulas may be used to calculate a reliability in recursively defined networks which are widely used in the solid state physics and in the nanotechnology.

### 5. Asymptotic analysis of bridge scheme

The simplest superposition of graphs is parallel-sequential graphs. But there are graphs widely used in the reliability theory, which are not parallel-sequential. One of them is a bridge scheme.

Consider the non-oriented graph \( \Gamma \) with the nodes set \( U = \{u_i, i = 0, \ldots, 3\} \) and with the edges set \( W = \{w_j, j = 1, \ldots, 5\} \), where

\[
w_1 = (u_0, u_1), \; w_2 = (u_0, u_2), \; w_3 = (u_1, u_3), \; w_4 = (u_2, u_3), \; w_5 = (u_1, u_2).
\]

The vertex \( u_0 \) is initial and the vertex \( u_3 \) is final. The edge \( w_3 \) is a bridge element in the graph \( \Gamma \). The graph \( \Gamma \) is called the bridge scheme in the reliability theory. Define the \( \Gamma_1 \) by a deleting of the edge \( w_5 \) from the graph \( \Gamma \). Introduce the graph \( \Gamma_2 \) by an aliasing of the nodes \( u_1, u_2 \) in the graph \( \Gamma_1 \).
Suppose that the edges $w_1, ..., w_5$ work independently and define positive numbers $c(w_i) = c_i, 1 \leq i \leq 5$

$$C_1 = \min \left( \max \left( c_1, c_3 \right), \max \left( c_2, c_4 \right) \right), C_2 = \max \left( \min \left( c_1, c_2 \right), \min \left( c_3, c_4 \right) \right), C_2 \leq C_1.$$  

If random logical variables $\beta_1, \beta_2$ characterize the nodes $u_0, u_3$ connectivity in the graphs $\Gamma, \Gamma_1, \Gamma_2$, correspondingly, then from the complete probability formula we have:

$$P(\beta = 1) = P_{w_5}(h) P(\beta_2 = 1) + \left(1 - P_{w_5}(h)\right) P(\beta_1 = 1), P(\beta_1 = 1) \leq P(\beta_2 = 1).$$ (11)

From the theorem 1 and the equalities (11) obtain the statement which characterizes a role of the bridge element.

**Theorem 9.** If $p_w(h) \sim \exp \left( -h^{-c(w)} \right), h \rightarrow 0$, where $c(w) > 0, w \in W$, then

$$-\ln P(\beta = 1) \sim h^{-D}, \quad D = \min \left( C_1, \max \left( C_2, C_3 \right) \right).$$ (12)

**References**